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Some remarks on the Gauss decomposition for quantum group $GL_q(n)$ with application to q-bosonization

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Abstract. The Gauss decomposition of quantum group $GL_q(n)$ is defined purely algebraically. The entries of the upper-triangular matrix commutes with those of the lower-triangular one. The quantum determinant is the product of the diagonal matrix elements. This decomposition gives rise to a new bosonization of the $GL_q(n)$ in terms of $2n$ q-oscillators.

In investigating algebraic systems it is often useful to handle their homomorphic realizations by means of simpler systems. The representation of Lie algebra generators by creation and annihilation operators of bosonic oscillators (bosonization) supplies us with a well known example. When we turn to the more complicated case of quantum groups and quantum algebras [1] it is natural to change the usual oscillators to deformed ones (q-oscillators) [2–6]. The q-bosonization procedure for quantum algebras was given in [5] with sufficient completeness. However, the situation for quantum groups is quite different. It seems first attempts to this end have been done for the quantum group $GL_q(n)$ only in the special case $q^n = 1$ [7–9]. In the general case, $q \in C\backslash\{0\}$, the examples of q-bosonization were given in [10] and [11] for $GL_q(2)$ and $GL_q(3)$, respectively. An attempt to generalize these results to $GL_q(n)$ was undertaken in a somewhat complicated fashion in [12]. The deficiency of this approach [10, 12] consists in the use of specific features of the Fock representation for the q-oscillators. As a result, a wide class of non-Fock representations (listed, for instance, in [6, 13, 14]) are excluded from consideration. We must also mention the interesting works [15, 16] concerned with a similar problem for the matrix pseudogroups $S_\mu U(n)$, which used a somewhat different form of q-oscillators.

In a previous paper [17] we showed that the Gauss decomposition [1] for $GL_q(2)$ suggests q-bosonization in a very simple and purely algebraic way. In this paper we shall make some remarks on the general properties of the $GL_q(n)$ Gauss decomposition and generalize to this quantum group the algebraic procedure of q-bosonization suggested earlier [17] for $GL_q(2)$. As an illustration of the results we shall consider the $GL_q(3)$ case in some detail. For the other application of the Gauss decomposition see in [22].

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Let us briefly recall some definitions and results of [1]. The quantum group $GL_q(n)$ is defined as an associative unital C-algebra which is generated by n^2 elements T_{ij} subject to the commutation relations

$$\begin{array}{ll} T_{ij}T_{ik} = qT_{ik}T_{ij} & T_{ik}T_{lj} = T_{lj}T_{ik} \\ T_{ik}T_{lk} = qT_{lk}T_{ik} & [T_{ij}, T_{lk}] = \lambda T_{ik}T_{lj} \end{array} \tag{1}$$

where $q \in C\backslash\{0\}$, $\lambda = q - q^{-1}$, $1 \leqslant j < k \leqslant n$, $1 \leqslant i < l \leqslant n$. In addition, $GL_q(n)$ is endowed with a Hopf algebra structure which introduced by three maps: a comultiplication Δ, a counit ϵ and an antipode S [1].

Let $T = (T_{ij})$ be n by n matrix of $GL_q(n)$ generators (q-matrix). From the commutation relations (1) it follows that the quantum (q-)determinant

$$D_q(T) \equiv \det_q T = \sum_\sigma (-q)^{l(\sigma)} \prod_{i=1}^{n} T_{i\sigma(i)} \tag{2}$$

belongs to the centre of the Hopf algebra $GL_q(n)$. In the definition (2) the sum is over all the permutations σ of the set $(1, 2, \ldots, n)$, and σ (l) is length of σ. There is additional assumption: $\det_q T \neq 0$. Supposing the invertibility of $D_q(T)$, one can calculate the two-sided inverse matrix T^{-1}

$$(T^{-1})_{ik} = (D_q(T))^{-1}(-q)^{i-k}D_q(M_{ik})$$

where $D_q(M_{ik})$ is a q-determinant of the minor matrix M_{ik} which is obtained from T by removing of the ith row and kth column. In the case of $GL_q(n)$ the above mentioned coalgebra maps are specified by the formulae

$$\Delta(T) = T(\otimes)T \qquad \epsilon(T) = 1 \qquad S(T) = T^{-1} \tag{3}$$

where 1 is the unit matrix and $(\otimes)$ refers to the usual matrix multiplication with a tensor product of matrix elements.

According to the R-matrix approach [1] the equation

$$RT_1T_2 = T_2T_1R \tag{4}$$

encodes the commutation relations between generators of a quantum group. In this equation R is a square number matrix of order n^2, T is a q-matrix and $T_1 = T \otimes 1$, $T_2 = 1 \otimes T$. The commutation relations (1) follow from (4) for the R-matrix corresponding to the Lie algebra $sl(n)$ [1]. The Lie group $GL(n)$ can be defined as an endomorphism group of complex n-dimensional linear space C^n. In complete analogy, the quantum group $GL_q(n)$ can be defined by its co-action δ on the related quantum space C_q^n. This view gives the basis of the non-commutative geometry approach to quantum deformations [18–20].

The Gauss decomposition for the $GL_q(n)$ q-matrix T is

$$T = (T_{ik}) = T_L T_D T_R$$

where $T_L = (u_{ik})$ and $T_R = (z_{ik})$ are respectively upper- and lower-triangular matrices with units at their diagonals, and $T_D = (A_{ik})$, with $A_{ik} = \delta_{ik}A_k$, is a diagonal matrix. It is not difficult to express the 'old' generators T_{ik} in terms of the 'new' ones, that is by the elements of the matrices T_L, T_D and T_R. For convenience we shall formulate the main properties of the Gauss decomposition as follows:

Proposition 1. The commutation relations (1) of the quantum group $GL_q(n)$ are fulfilled if the following commutation relations for the elements of the T_L, T_D and T_R matrices hold:

$$A_k u_{ij} = q^{\delta_{ik}-\delta_{jk}} u_{ij} A_k \qquad 1 \leqslant i, j, k \leqslant n \tag{5a}$$

$$u_{ik}u_{jk} = q^{\delta_{ik}-\delta_{jk}+1} u_{jk}u_{ik} \qquad 1 \leqslant k \leqslant n,\ i < j \tag{5b}$$

$$u_{ik}u_{il} = q^{\delta_{ij}-\delta_{ik}+1} u_{ij}u_{ik} \qquad 1 \leqslant i \leqslant n,\ k < j \tag{5c}$$

$$u_{ik}u_{lj} = q^{\delta_{il}-\delta_{kl}} u_{lj}u_{ik} \qquad i < l,\ k > j \tag{5d}$$

$$u_{ik}u_{lj} - q^{\delta_{ij}-\delta_{kl}} u_{lj}u_{ik} = \lambda q^{\delta_{ik}-\delta_{kl}} u_{lk}u_{ij} \qquad i < l,\ k < j \tag{5e}$$

$$\left[u_{ik}, z_{jl}\right] = [A_i, A_k] = 0 \qquad 1 \leqslant i, j, k, l \leqslant n \tag{5f}$$

$$A_k z_{ij} = q^{\delta_{kj}-\delta_{ik}} z_{ij} A_k \qquad 1 \leqslant i, j, k \leqslant n \tag{5g}$$

$$z_{ik}z_{jk} = q^{\delta_{jk}-\delta_{ik}+1} z_{jk}z_{ik} \qquad 1 \leqslant k \leqslant n,\ i < j \tag{5h}$$

$$z_{ik}z_{ij} = q^{\delta_{ik}-\delta_{ij}+1} z_{ij}z_{ik} \qquad 1 \leqslant i \leqslant n,\ k < j \tag{5i}$$

$$z_{ik}z_{lj} = q^{\delta_{kl}-\delta_{ij}} z_{lj}z_{ik} \qquad i < l,\ k > j \tag{5j}$$

$$z_{ik}z_{lj} - q^{\delta_{kl}-\delta_{ij}} z_{lj}z_{ik} = \lambda q^{\delta_{kl}-\delta_{ik}} z_{lk}z_{ij} \qquad i < l,\ k < j. \tag{5k}$$

Proposition 2. In terms of the 'new' generators the quantum determinant $D_q(T) = \det_q T$ has the simple form

$$D_q(T) = \det T_{\rm D} = A_{11}A_{22} \cdot \ldots \cdot A_{nn} \tag{6}$$

and commutes with every element of the matrices $T_{\rm L}$, $T_{\rm D}$ and $T_{\rm R}$.

(Expression (6) for the $GL_q(2)$ case was given in [19]). These two propositions can be checked by direct calculation.

Let us denote $\widetilde{T} = T_{\rm D}T_{\rm R}$, and $\widehat{T} = T_{\rm L}T_{\rm D}$. We would like to stress that the elements of each of the matrices ($T_{\rm L}$, $T_{\rm D}$, $T_{\rm R}$, $\widetilde{T}$, and $\widehat{T}$) form a set closed under the commutation relations (5) and, thus, define a deformed algebra.

Proposition 3. (a) The commutation relations between the elements of the matrix $\widetilde{T}$ (and $\widehat{T}$) are determined by the quantum group equation (4) with the same R-matrix as for $GL_q(n)$; (b) There is no R-matrix that gives the commutation relations between the elements of the matrix $T_{\rm R}$ (and $T_{\rm L}$) in the form of (4).

The first part of this proposition can be proved by direct verification. Any attempt to calculate an appropriate R-matrix from equation (4) in the case of the matrices $T_{\rm R}$ and $T_{\rm L}$ inevitably leads to contradictions. So the algebras defined by the elements of matrices $T_{\rm R}$ and $T_{\rm L}$, supply us with examples of non-R-matrix quantum deformations.

Using the homomorphic property of a comultiplication in $GL_q(n)$ we can define the comultiplications in the algebras connected with each of the matrices $T_{\rm R}$, $T_{\rm D}$, $T_{\rm L}$, $\widetilde{T}$ and $\widehat{T}$. (This was done for $GL_q(2)$ in [21]). However, such an inherited comultiplication has a *cumbersome form even for* $n = 2, 3$. *Nevertheless, there is a natural standard* (3) *Hopf* algebra structure [1] for the algebra generated by the 'new' generators.

Proposition 4. (a) The algebra generated by the elements of the matrix $\widetilde{T}$ is a Hopf algebra (under the co-operations defined in (3)) and gives us an example of a new quantum group. The same is true for $\widehat{T}$.
(b) The algebra generated by the elements of the matrices $T_{\rm R}, T_{\rm D}, T_{\rm L}$ is a Hopf algebra under the co-operations defined by the following formulae:

$$\Delta(A_i) = A_i \otimes A_i \qquad \Delta(u_{ij}) = \sum_k u_{ik}A_kA_j^{-1} \otimes u_{kj} \qquad \Delta(z_{ij}) = \sum_k z_{ik} \otimes A_i^{-1}A_k z_{kj}$$

$$\epsilon(u_{ij}) = \epsilon(z_{ji}) = 0 \quad i > j \qquad \epsilon(A_k) = 1$$

$$S(A_k) = A_k^{-1} \qquad S(u_{ij}) = A_i^{-1} A_j (u^{-1})_{ij} \qquad S(z_{ij}) = (z^{-1})_{ij} A_i A_j^{-1}.$$

Proposition 5. The map

$$\delta(X) = T(\otimes)X \qquad \delta(x_i) = \sum_{k=1}^{n} T_{ik} \otimes x_k$$

defines the co-action of the quantum groups $\widetilde{T}$ and $\widehat{T}$ on the quantum space C_q^n. Moreover, the maps $\widetilde{\delta} : C_q^n \to \widetilde{T}(\otimes)C_q^n$ and $\widehat{\delta} : C_q^n \to \widehat{T}(\otimes)C_q^n$ are algebra homomorphisms and endow C_q^n with a left $\widetilde{T}$- and $\widehat{T}$-comodule structure, respectively.

The proof of these two propositions is evident in view of the propositions given above.

Let $\{a_i^\dagger, a_i, N_i\}$ and $\{b_i^\dagger, b_i, M_i\}$, $i, j = 1 \div n$ be two independent families of mutually commuting q^{-1}- and q-oscillators [2–6] defined by the relations

$$a_i a_i^\dagger - q^{-1} a_i^\dagger a_i = q^{N_i} \qquad N_i a_i^\dagger = a_i^\dagger (N_i + 1) \qquad N_i a_i = a_i (N_i - 1) \tag{7}$$

$$b_i b_i^\dagger - q b_i^\dagger b_i = q^{-M_i} \qquad M_i b_i^\dagger = b_i^\dagger (M_i + 1) \qquad M_i b_i = b_i (M_i - 1). \tag{8}$$

Define

$$u_{ik} = \begin{cases} 0 & \text{if } i > k \\ 1 & \text{if } i = k \\ f_{ik} q^{N_{ik}} a_i^\dagger a_k & \text{if } i < k \end{cases} \qquad z_{ik} = \begin{cases} 0 & \text{if } i < k \\ 1 & \text{if } i = k \\ g_{ik} q^{-M_{ik}} b_i^\dagger b_k & \text{if } i > k \end{cases} \tag{9}$$

$$A_{ik} = \delta_{ik} q^{N_k - M_k} \tag{10}$$

where

$$N_{ik} = \sum_{j=i+1}^{k-1} N_j \qquad M_{ik} = \sum_{j=k+1}^{i-1} M_j$$

and $N_{ik} = 0$ $(M_{ik} = 0)$ if $i > (k-2)$, $(i < (k+2))$. Then we have

$$\det_q T = \prod_{i=1}^{n} A_{ii} = q^{N-M} \qquad N = \sum_{j=1}^{n} N_j \qquad M = \sum_{j=1}^{n} M_j.$$

Proposition 6. Expressions (9), (10) satisfy the commutation relations (5) if the number coefficients f_{ik} and g_{ik} fulfil the equations

$$\begin{cases} f_{ij} f_{kl} = f_{kj} f_{il} & i < k < j < l \\ f_{ij} f_{jk} = \lambda q^{-1} f_{ik} & i < j < k \end{cases}$$
$$\begin{cases} g_{ij} g_{kl} = g_{kj} g_{il} & k > i > l > j \\ g_{ij} g_{ki} = -q \lambda g_{kj} & k > j > i. \end{cases} \tag{11}$$

In this case equations (9), (10) realize q-bosonization of the quantum groups T_{R}, T_{D}, T_{L} and, consequently $\widetilde{T}$, $\widehat{T}$, $GL_q(n)$.

There are several solutions of equations (11). The simplest one is

$$f_{ij} = \lambda/q \qquad g_{ij} = -q\lambda. \tag{12}$$

It is worth noting that in the present version of $GL_q(n)$ q-bosonization $2n$ independent deformed oscillators are used. On the other hand, following the method of [10–12] for this purpose requires $n(n-1)/2$ q-oscillators.

Let us remark that for q-bosonization one can use one type of q-oscillators only but in this case the expressions are not so symmetric.

To illustrate the above results, we consider the quantum group $GL_q(3)$ as an example. The case of $GL_q(2)$ studied in [17], is too simple because both matrices T_{R} and T_{L} contain only one non-trivial element. The Gauss decomposition for $GL_q(3)$ has the form

$$T = T_{\mathrm{L}}T_{\mathrm{D}}T_{\mathrm{R}} = \begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix}$$

$$= \begin{pmatrix} A + uBx + vCy & uB + vCz & vC \\ Bx + wCy & B + wCz & wC \\ Cy & Cz & C \end{pmatrix}$$

$$\widetilde{T} = T_{\mathrm{D}}T_{\mathrm{R}} = \begin{pmatrix} A & 0 & 0 \\ Bx & B & 0 \\ Cy & Cz & C \end{pmatrix} \qquad \widehat{T} = T_{\mathrm{L}}T_{\mathrm{D}} = \begin{pmatrix} A & uB & vC \\ 0 & B & wC \\ 0 & 0 & C \end{pmatrix}.$$

The following non-trivial commutation relations from (5) for the 'new' generators are

$$\begin{array}{lll} vw = qwv & uv = qvu & quw - wu = \lambda v \\ xy = qyx & yz = qzy & xz - q^{-1}zx = \lambda y \\ uB = qBu & uC = Cu & Au = quA \\ vC = qCv & vB = Bv & Av = qvA \\ wC = qCw & wA = wA & Bw = qwB \\ xB = qBx & xC = Cx & Ax = qxA \\ yC = qCy & yB = By & Ay = qyA \\ zC = qCz & zA = Az & Bz = qzB. \end{array} \tag{13}$$

The quantum determinant $D_q(T) = ABC$ commutes with all of the 'new' generators and, hence, with $\widetilde{T}$ and $\widehat{T}$.

Applying the usual comultiplication $\Delta(T) = T(\otimes)T$ in $GL_q(3)$ and denoting

$$Q = y \otimes v + z \otimes w + 1 \otimes 1 \equiv y_1 v_2 + z_1 w_2 + 1$$

$$E = 1 + x_1 u_2 - (x_1 v_2 + w_2)Q^{-1}(y_1 u_2 + z_1)$$

$$K = (1 - u_2 E^{-1} L - v_2 Q^{-1}[y_1 - (y_1 u_2 + z_1)E^{-1}L])$$

where $L = [x_1 - (x_1 v_2 + w_2)Q^{-1}y_1]$, one can obtain the following expressions for inheritable comultiplication:

$$\Delta(A) = A_1 K A_2 \qquad \Delta(B) = B_1 E B_2 \qquad \Delta(C) = C_1 Q C_2$$

$$\Delta(u) = u_1 + A_1 u_2 (B_1 E)^{-1} - A_1 v_2 Q^{-1}(y_1 u_2 + z_1)(B_1 E)^{-1}$$

$$\Delta(v) = v_1 + (A_1 v_2 + u_1 B_1 (x_1 v_2 + w_2))Q^{-1} C_1^{-1}$$

$$\Delta(w) = w_1 + B_1(x_1 v_2 + w_2) Q^{-1} C_1^{-1}$$
$$\Delta(x) = x_2 + (EB_2)^{-1}[x_1 - (x_1 v_2 + w_2) Q^{-1} y_1] A_2$$
$$\Delta(y) = y_2 + C_2^{-1} Q^{-1}[y_1 A_2 + (y_1 u_2 + z_1) B_2 x_2]$$
$$\Delta(z) = C_2^{-1} Q^{-1}(y_1 u_2 + z_1) B_2 + z_2.$$

We recall that $A_1 = A \otimes 1$, $A_2 = 1 \otimes A$, etc. As was pointed out above, there is a 'natural' comultiplication, together with 'inheritable' one, in $\widetilde{T}$ and $\widehat{T}$. These two maps are not equivalent because they give obviously different expression for the element $C = T_{33}$. Therefore, we are dealing with new quantum groups.

For the quantum group generated by the elements of the matrices T_R, T_D, T_L the natural Hopf algebra structure defined in proposition 4(b) is given by the following expressions:

$$\Delta(A) = A \otimes A \qquad \Delta(B) = B \otimes B \qquad \Delta(C) = C \otimes C \tag{14}$$

$$\begin{aligned} \Delta(x) &= x \otimes B^{-1}A + 1 \otimes x \\ \Delta(y) &= y \otimes C^{-1}A + z \otimes C^{-1}Bx + 1 \otimes y \\ \Delta(z) &= z \otimes C^{-1}B + 1 \otimes z \end{aligned} \tag{15}$$

$$\begin{aligned} \Delta(u) &= AB^{-1} \otimes u + u \otimes 1 \\ \Delta(v) &= AC^{-1} \otimes v + uBC^{-1} \otimes w + v \otimes 1 \\ \Delta(w) &= BC^{-1} \otimes w + w \otimes 1. \end{aligned} \tag{16}$$

It is not difficult to check that the matrices

$$\widetilde{T}^{-1} = \begin{pmatrix} A^{-1} & 0 & 0 \\ -qA^{-1}x & B^{-1} & 0 \\ q^2 A^{-1}(xz - qy) & -qB^{-1}z & C^{-1} \end{pmatrix}$$

$$\widehat{T}^{-1} = \begin{pmatrix} A^{-1} & -quA^{-1} & q^{-1}(uw - v)A^{-1} \\ 0 & B^{-1} & -q^{-1}wB^{-1} \\ 0 & 0 & C^{-1} \end{pmatrix}$$

are the two-sided inverse matrices of the $\widetilde{T}$ and $\widehat{T}$, respectively. Therefore, the antipodes and counits can be determined according to (3). At the element level one has

$$S(A) = A^{-1} \qquad S(B) = B^{-1} \qquad S(C) = C^{-1}$$
$$S(x) = -q^2 A^{-1}Bx = -xA^{-1}B$$
$$S(z) = -q^2 B^{-1}Cz = -zB^{-1}C$$
$$S(y) = q^3 A^{-1}C(xz - qy) = (zx - y)A^{-1}C$$
$$S(u) = -q^{-1}BuA^{-1} = -BA^{-1}u$$
$$S(w) = -q^{-1}CwB^{-1} = -CB^{-1}w$$
$$S(v) = q^{-1}C(uw - v)A^{-1} = CA^{-1}(uw - v)$$

$$\varepsilon(A) = \varepsilon(B) = \varepsilon(C) = 1 \qquad \varepsilon(u) = \varepsilon(v) = \varepsilon(w) = 0 \qquad \varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 0.$$

With these definitions the Hopf algebra axioms [1] are satisfied. As a result, the algebras associated with $\widetilde{T}$ and $\widehat{T}$ matrices are endowed with a Hopf algebra structure. Therefore, they can indeed be considered as new quantum groups.

In conclusion we write down the q-bosonization of the $GL_q(3)$ using the solution (12) of the equations (11). So, taking the three q^{-1}-oscillators (7) and the three q-oscillators (8), and using (9), (10), one gets

$$u = \lambda q^{-1} a_1^\dagger a_2 \qquad v = \lambda q^{-1} q^{N_2} a_1^\dagger a_3 \qquad w = \lambda q^{-1} a_2^\dagger a_3$$
$$x = -\lambda q b_2^\dagger b_1 \qquad y = -\lambda q q^{-M_2} b_3^\dagger b_1 \qquad z = -\lambda q b_3^\dagger b_2$$
$$A = q^{N_1-M_1} \qquad B = q^{N_2-M_2} \qquad A = q^{N_3-M_3}.$$

For the original $GL_q(3)$-generators this gives

$$T_{11} = q^{N_1-M_1} - q\lambda^2 q^{N_2-M_2}\left[a_1^\dagger a_2 b_2^\dagger b_1 + q^{N_3-M_3} a_1^\dagger a_3 b_3^\dagger b_1\right]$$
$$T_{12} = \lambda q^{-1}\left[a_1^\dagger a_2 q^{N_2-M_2} - \lambda q q^{N_2+N_3-M_3} a_1^\dagger a_3 b_3^\dagger b_2\right]$$
$$T_{13} = \lambda q^{-1} a_1^\dagger a_3 q^{N_2+N_3-M_3}$$
$$T_{21} = -\lambda q\left[q^{N_2-M_2} b_2^\dagger b_1 + \lambda q^{N_3-M_3-M_2} a_2^\dagger a_3 b_3^\dagger b_1\right]$$
$$T_{22} = q^{N_2-M_2} - \lambda^2 q^{N_3-M_3} a_2^\dagger a_3 b_3^\dagger b_2$$
$$T_{23} = \lambda q^{N_3-M_3} a_2^\dagger a_3$$
$$T_{31} = -\lambda q q^{N_3-M_2-M_3} b_3^\dagger a_1 \qquad T_{32} = -\lambda q q^{N_3-M_3} b_3^\dagger b_2 \qquad T_{33} = q^{N_3-M_3}.$$

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